Chapter 8

Mathematical Development

Brian Butterworth and Sashank Varma

Introduction

The importance of mathematics instruction has been stressed, quite rightly, in many official reports in the United Kingdom, the United States, and other nations. Napoleon famously said that mathematics is "intimately connected with the prosperity of the state". In his foreword to the Cockcroft report on math teaching in 1982, Sir Keith Joseph, Secretary of State for Education and Science, wrote "Few subjects are as important to the future of the nation as mathematics" (Cockcroft, 1982). Since Cockcroft, in the United Kingdom alone, there has been Professor Adrian Smith's report on post-14 math (Smith, 2004), and Sir Peter Williams' report on primary math (Williams, 2008). Similarly, the US National Research Council (National Research Council Committee on Early Childhood Mathematics, 2009) noted that "The new demands of international competition in the 21st century require a workforce that is competent in and comfortable with mathematics;" and to that end "The committee [of experts] was charged with examining existing research in order to develop appropriate mathematics learning objectives for preschool children; providing evidence-based insights related to curriculum, instruction, and teacher education for achieving these learning objectives" (p. 1). In 2010, the OECD's report, *The High Cost of Low Educational Performance*, demonstrated that the standard of math drives GDP growth: the standard in 1960 was a good predictor of GDP growth up to 2000; and the improvement in educational standard from 1975 to 2000 was

Educational Neuroscience, First Edition. Edited by Denis Mareschal, Brian Butterworth and Andy Tolmie.

highly correlated with improvement in GDP growth. In particular, the report looked at the potential effects of improving standards in math. So, for example, they found that if the UK improved the standard of the 11% of children who failed to reach the PISA minimum level (which is not very high) to the minimum level, then the effect on GDP growth would be about 0.44%. Not much, you might think, but with an average rate of GDP growth of 1.5%, this would be a massive and cumulative increase of nearly one-third.

Poor math has consequences for the lives of individuals. A UK survey found that learners with poor math are more likely to be unemployed, depressed, and in trouble with the law (Parsons & Bynner, 2005). The accountancy firm, KPMG, estimated that the cost to the United Kingdom of poor math in terms of lost direct and indirect taxes, unemployment benefits, justice costs, and additional educational costs was £2.4 *billion* per year (Gross, Hudson, & Price, 2009).

Can educational neuroscience make a contribution to improving society and the lives of individuals by improving math education?

There is now an extensive psychological and neuroscience literature on mathematical cognition and its development. This chapter focuses on studies of number understanding, as this topic has received the greatest attention in the literature in terms both of cognition and of its neural basis, and is most relevant to the problems of education. As in the other chapters of this book, development is considered across four phases: infancy and childhood (ages 0–5), primary and middle school (5–12), secondary school and adolescence (12–18), and adulthood.

Neural roadmap

Before beginning, we provide a roadmap of the brain areas of greatest relevance to mathematical thinking, and introduce their putative cognitive functions. We know about their roles from two main sources. The earliest and still influential source is the effect of *brain damage*: how does damage to area A affect different aspects of mathematical processing? Of course, it is vital to compare the effects of damage to area A with damage to other areas of the brain. More recently, it has been possible to induce transient neural malfunctions using *transcranial magnetic stimulation* on normal brains. The other source of information is the map of brain activity when the healthy brain is carrying out a mathematical task. Mapping using *functional magnetic resonance imaging* (fMRI) gives reasonable localization, but not much information about the timecourse of the cognitive processes, since it measures changes in blood flow, which responds slowly to the activity of the brain cells. By contrast, *electroencephalography*, which records changes in electrical potentials across the scalp, gives good temporal resolution, but poor spatial resolution. (See Chapter 2.)

Three brain areas in the parietal lobes are particularly important for numbers and arithmetic (Dehaene, Piazza, Pinel, & Cohen, 2003). (1) *The intraparietal sulcus (IPS)* is the neural correlate of the *magnitude representations* that *number symbols* denote. (Since this sulcus is long, the *horizontal* middle section – hIPS – appears most relevant.) Both left and right IPS are active in most numerical tasks. (2) *The left angular gyrus (AG)* is involved in retrieval of previously learned number facts (see especially Delazer et al., 2005; Ischebeck et al., 2006). When the left AG is damaged, calculation can be severely affected. (3) *The posterior superior parietal lobule (SPL)* is one of the areas involved in relating numbers to space, for example, in counting visible objects.

Other brain areas also play important roles in mathematical cognition and development. For example, the right fusiform gyrus (rFG) is associated with processing the visual form of mathematical symbols (Rykhlevskaia, Uddin, Kondos, & Menon, 2009). The right inferior frontal gyrus (rIFG) is implicated in spatial working memory, and in phenomena that link numbers to space (Rusconi, Bueti, Walsh, & Butterworth, 2011). For more abstract mathematical thinking, the prefrontal cortex is important. When it is damaged, routine or previously learned problems can be solved, but novel problems cannot (Shallice & Evans, 1978).

These various brain areas and their putative roles in mathematical cognition are depicted in Figure 8.1. We refer to this figure later to situate the discussion of individual studies.

Theoretical roadmap

As noted earlier, most neuroscience research has focused on numbers and arithmetic, which, in terms of the curriculum and everyday life, are the most important aspects of mathematics. It is therefore important to be clear about what numbers are and what we know about how they are represented and processed in the brain.

In our numerate society numbers are used in many different ways. Here our focus is on numbers as abstract properties of sets, for example, to characterize the number of fingers on a hand, the number of dwarves with Snow White, or the number of wishes given by a genie. These are *cardinal* numbers, sometimes called *numerosities*, and are ordered by *magnitude*. So five is larger than four, and a set of five will include a set of four. Two sets have the same numerosity – are exactly equal – when the members of one can be put in one-to-one correspondence with members of the other. This means that adding a number to a set or subtracting a member from a set will affect the numerosity of the set. This is the use or meaning of number that is relevant to arithmetic. There is some disagreement

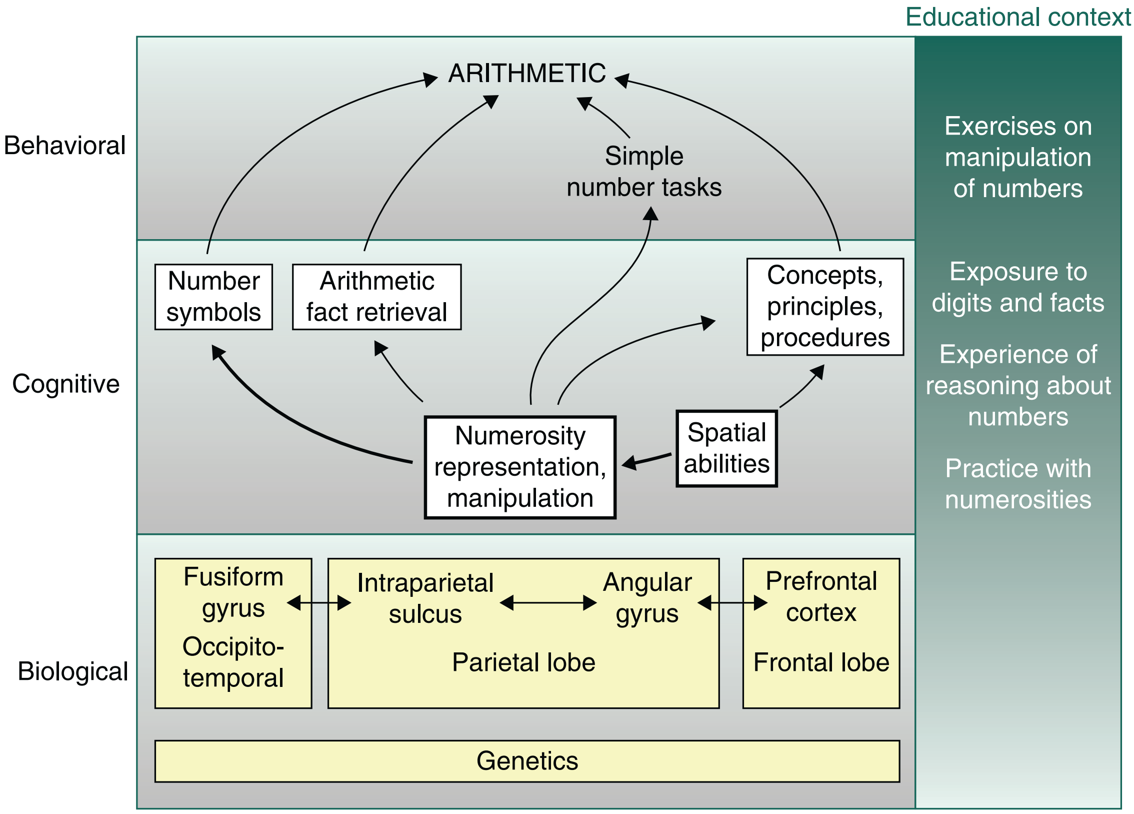


Figure 8.1 Causal model of possible inter-relations between biological, cognitive, and simple behavioral levels. Here, the only environmental factors we address are educational. If parietal areas, especially the IPS, fail to develop normally, there will be an impairment at the cognitive level in numerosity representation and consequential impairments for other relevant cognitive systems revealed in behavioral abnormalities. The link between the occipitotemporal and parietal cortex is required for mapping number symbols (digits and number words) to numerosity representations. The prefrontal cortex supports learning new facts and procedures. The multiple levels of the theory suggest the instructional interventions on which educational scientists should focus. (From Butterworth, Varma, and Laurillard (2011) with permission.)

about how magnitudes are represented in the brain, and how we come to have representations of exactly five.

Another familiar use for number is to order things – such as the pages of this book. Page 100 does not have a larger magnitude than page 99 (though the set of pages to 100 will have a larger magnitude than the set to 99, of course). Logicians and mathematicians call these *ordinal* numbers, or *ranks*, and they are sometimes, but not always, referred to by separate vocabulary items: *first*, *second*, *third*, 1st, 2nd, 3rd – though usually the same words and symbols are used for ordinals and numerosities, as in page or house numbers. There is now some evidence that the neural representation of ordinals is distinct from that of cardinals (Delazer & Butterworth, 1997).

Still another use of numbers is as *labels*. For example, in bus numbers, telephone numbers, bar codes, TV channels, neither the magnitude of the number nor its order in a sequence is relevant. Thus it makes no sense to say that John's telephone number is larger than Jim's, or that it comes after Jim's.

One of the problems for the learner is to distinguish these uses of number, and ensure that there is the correct mapping between the number symbol – the word or the digit – and its appropriate referent. This is particularly important since understanding the symbol systems is a key to talking and learning about numbers and arithmetic both in school and out of it. Manipulation of symbols is also a mentally efficient way of manipulating and storing arithmetical concepts. As the philosopher A. N. Whitehead observed, an understanding of symbolic notation relieves "the brain of all unnecessary work … and sets it free to concentrate on more advanced problems" (Whitehead, 1948).

Two important effects

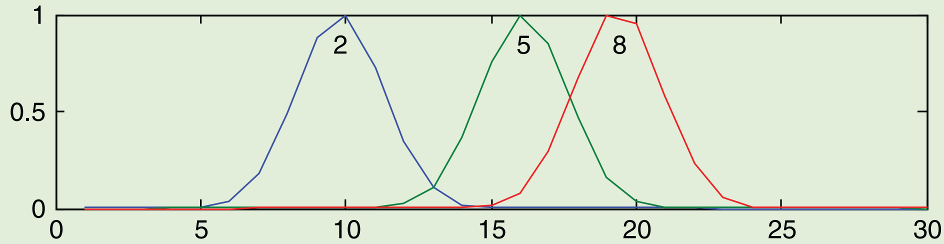
When people are working with symbolic numbers, it is often important to know how they are interpreting them. In particular, when they are engaged in a standard arithmetical task, are representations of numerical magnitude elicited? Two effects are standardly used to test for this.

The first is the *distance effect* – the seminal finding that, when comparing two numbers (i.e., judging which one is greater or lesser), the larger the difference in magnitude, the shorter the response time (Moyer & Landauer, 1967), suggesting that magnitude representations are being compared. The distance effect observed when comparing two symbolic numbers will be referred to as the *symbolic* distance effect, and the one observed when comparing the numerosities of two sets of objects will be referred to as the *nonsymbolic* distance effect (Buckley & Gillman, 1974). It is important to distinguish these two tasks because it is always possible that a learner may be able to do one normally, but not the other. For example, if the learner can do the nonsymbolic task in the normal way, where magnitudes are directly represented in the stimulus, but not the symbolic task, this could imply a problem in linking the symbol to its magnitude representation.

The other diagnostic effect is the *problem size effect*, or really just the size effect. Responses are slower and less accurate when the numbers are larger. This is so reliable, that one famous paper by Zbrodoff and Logan (2005) is titled "What everyone finds: The problem size effect." It may seem surprising that the time it takes to solve even single-digit additions or multiplication table facts depends on the size of the numbers, thus it takes longer to solve $9+8$ and 9×8 than $6+7$ and 6×7, even though these facts are highly overlearned for

Box 8.1 *Approximate numerosities.*

It has been suggested that arithmetical abilities are built on an inherited system for representing numerosities in an approximate way. So instead of representing fiveness exactly, it is represented approximately, and mapped onto an analog magnitude representation, usually with compression. This is represented pictorially here, where the horizontal scale is an arbitrary linear scale, and the vertical scale represents idealized activation, with the peak of activation representing the most probable response. In this model, the representation of each number overlaps with other numbers.



Several studies have found a correlation between arithmetical abilities and measures of the ability to discriminate numerosities greater than about six (Halberda, Mazzocco, & Feigenson, 2008; Piazza et al., 2010), though some have failed to find this (Iuculano, Tang, Hall, & Butterworth, 2008). Notice that the representation for each numerosity overlaps with that of its neighbors. This means that the basic numerical and arithmetical operations cannot be carried out on these representations. For example, the numerical equivalence between two representations cannot be established, whereas between sets it can be established by showing one-to-one correspondence between members of each set. Transformations that affect numerosity, such as adding or subtracting an element, cannot be determined with these analog representations. Children as young as three, who cannot yet count, notice which transformations affect numerosity (Sarnecka & Gelman, 2004). It has been argued by Butterworth (2010) that these representations cannot be foundational for arithmetic: representations of sets are necessary.

most numerate adults. Now, if a learner does not show this effect, it could also be diagnostic of an important individual difference. For example, the learner may be able to recall a fact through rote learning but not really understand it because he or she is not evoking the number magnitudes.

Mathematical Development

The neural background to all cognitive development is the way the brain changes from conception to old age. Chapter 2 reviewed some of the evidence relevant to the development of mathematics. First, the brain at birth is not the same as the brain later in life. The infant brain has more brain cells (neurons) and more connections (synapses) than the adult brain, and this means that it is more "plastic," that is, more responsive to experience, including formal and informal instruction, than later on. As a consequence, learning new material appears to be easier earlier than later. Nevertheless, the newborn brain is not a blank slate. It comes equipped with structures and biases that support learning, especially learning concepts that evolution has found important. Numerosities are among these concepts.

Infancy and childhood (0–5)

The inherited capacity to represent and discriminate stimuli on the basis of their numerosity has been observed in infants. This has required developmental psychologists to construct tasks that do not require verbal responses, since one cannot ask infants how many objects they are looking at. Many studies have used a *dishabituation* paradigm, which capitalizes on the fact that young infants will look longer at a stimulus if it differs from prior stimuli in a meaningful way. The first study to show this using a dishabituation paradigm established that at 5 months old babies are sensitive to changes in small numerosities in visual displays of two to six objects (Starkey & Cooper, 1980); subsequent research showed that this sensitivity was present even in the first week of life (Antell & Keating, 1983). Six-month-olds dishabituate to displays of 8 versus 16 objects (a 2.0 ratio) but not 8 versus 12 objects (a l.5 ratio) (Xu & Spelke, 2000).

A similar paradigm has been used to establish the neural response to changes in numerosity using event-related potentials (electroencephalography) from three-month-old infants while they were presented with a continuous stream of images, each showing a set of identical objects (Izard, Dehaene-Lambertz, & Dehaene, 2008). These were the habituation images. Within a given run, most sets had the same numerosity ("standard number") and the same object, but occasionally test images that could differ from the habituation images in number and/or object identity. It was thus possible to compare the visual event-related potentials evoked by unforeseen changes (dishabituation) either in the numerosity of a set ("deviant number") or the identity of objects forming the set. Three numerosity contrasts were investigated: two versus three, four versus eight, and four versus 12.

They found that all these numerosity contrasts produced a dishabituation effect – a difference in the pattern of evoked potentials; and the effect for changes in numerosity was different from the effect of change in the object. So it was not just *change* that the infant brain was responding to: there was a specific effect for *change in numerosity*. In particular, there was a significant response in the right parietal lobe, an area that is involved in numerosity processing in adults (see, e.g., Castelli, Glaser, & Butterworth, 2006; Piazza, Mechelli, Price, & Butterworth, 2006; Vetter, Butterworth, & Bahrami, 2011). Thus, not only is the cognitive capacity to discriminate numerosities present in infants, the neural mechanism that grows into adult competence is already in place.

Further evidence for this comes from a study of four-year-olds using a very similar habituation paradigm, but with the neural response measured by fMRI, which can give a more precise localization (Cantlon, Brannon, Carter, & Pelphrey, 2006). (This method is sometimes called "fMRI adaptation;" see Chapter 2). This study had four-year-old children watch sequences of visual stimuli. There were sequences where the same object (e.g., a circle) appeared in the same numerosity (e.g., 16 items), followed by a display with either a different object (e.g., a square) or a different numerosity (e.g., 32 items). The question was which brain areas would show a response, or neurally dishabituate, to numerical deviants versus object deviants. Children showed increasing activation to numerical deviants in right IPS, the same area that Izard et al. found in three-month-olds. The adults in this study showed increased activation in the same area, but also in the left IPS, suggesting that by adulthood these very simple numerosity representations have been connected with left-hemisphere functions, including language.

Primary and middle childhood (5–12)

The distance effect has been used to investigate the natural number representations of primary and middle school children when comparing sets of objects and digits. For example, in comparing sets of objects, six-, seven-, and eight-year old children show a nonsymbolic distance effect (Ansari & Dhital, 2006). Landerl and Kölle (2009) similarly found a nonsymbolic distance effect in eight-, nine-, and 10-year old children. Developmentally, people become more accurate at comparing numerosities as they get older (Piazza et al., 2010).

Beginning in primary school, it becomes possible to investigate children's understanding of the digits. Sekuler and Mierkiewicz (1977) had kindergarteners, first graders, fourth graders, seventh graders, and adults compare pairs of numbers. They found a symbolic distance effect at all ages. This finding has been replicated and extended with six-, seven-, and eight-year-old children (Holloway & Ansari, 2009) and with eight-, nine-, and 10-year old children (Landerl & Kölle, 2009).

Ansari and Dhital (2006) investigated the neural correlates of numerosity in 10-year-old children and adults. The children displayed a *neural distance effect* – greater activation for near-distance comparisons than far-distance comparisons – in left IPS. The adults also displayed a neural distance effect in left IPS, as well as one in right IPS. These results are further evidence of comparable magnitude representations in young children and adults, with similar neural bases (Cantlon et al., 2006). The neural correlates of the symbolic distance effect have been investigated in 10-year-old children (Ansari, Garcia, Lucas, Hamon, & Dhital, 2005). The adults in this study displayed a neural distance effect in bilateral IPS, a finding that has been replicated in numerous studies. By contrast, the children displayed neural distance effects in a network of right prefrontal areas, including right IFG.

Taken together, these results demonstrate both continuities and discontinuities in the development of number. Young infants and primary- and middle-school children show nonsymbolic distance effects that are comparable to those of adults, both behaviorally and neurally, suggesting a common representation of numerosity in IPS. The story for the symbolic distance effect is different: while both adults and children show this effect behaviorally, the neural correlates are different, with adults showing modulated activation in both the left and right IPS but 10-year-old children showing it in prefrontal cortex. Thus, the neuroscience data reveal a developmental discontinuity in how number symbols are processed.

This is educationally important, since it has been shown that distance effects, both nonsymbolic (Halberda et al., 2008) and symbolic (Holloway & Ansari, 2009) are correlated with arithmetical performance.

Lifelong learning (adulthood)

Nonsymbolic distance effects are regularly found in adults (Buckley & Gillman, 1974). There is also a symbolic distance effect (Buckley & Gillman, 1974; Moyer & Landauer, 1967). The assumption has been that both effects reflect a common magnitude representation. This assumption has been corroborated by recent studies investigating the neural correlates of these effects. In general, close numbers evoke more activation than distant numbers in the bilateral IPS, whether comparing numerosities (Ansari & Dhital, 2006; Castelli et al., 2006) or number symbols (see, e.g., Fias, Lammertyn, Reynvoet, Dupont, & Orban, 2003; Pinel, Dehaene, Rivière, & Le Bihan, 2001).

There is also an interesting developmental shift from primary and middle school to adulthood in understanding natural number symbols. In children, the neural correlate of the symbolic distance effect is right prefrontal cortex (Ansari et al., 2005), which is associated with executive function and controlled cognition more generally (Shallice & Evans, 1978). For them, mapping number symbols to magnitude referents is an effortful process. By contrast, the neural correlates for adults include bilateral IPS, indicating a more integrated neural representation of number symbols and their magnitude referents.

A strikingly similar anterior-to-posterior shift in the brain is observed in arithmetic development. Rivera, Reiss, Eckert, and Menon (2005) had children aged 8–19 solve addition and subtraction problems. Behaviorally, accuracy rates were high across the age range (greater than 85%), indicating that all of the children were arithmetically competent. Response times decreased linearly with age, from over 2000 ms in the younger children to close to 1000 ms in the older children. It is interesting to consider the neural implications of this behavioral speedup, which has been found in other studies (e.g., Koshmider & Ashcraft, 1991). One possibility is that children across the age range activate the same cortical network, i.e., use similar processing, but that younger children are less efficient than older children. In fact, the fMRI data revealed a very different picture. Younger children activated a network of frontal areas associated with executive function, including bilateral middle frontal gyrus (MFG) and left anterior cingulate cortex (ACC). They also activated frontal and temporal areas associated with controlled retrieval from declarative memory, including left IFG and medial temporal lobe/hippocampus. In other words, younger children activated a cortical network associated with effortful or strategic processing to solve arithmetic problems. Over development, activation in this strategic network decreased and activation in a more posterior and more domain-specific network increased. This included left FG, which is associated with visual symbol processing; left supramarginal gyrus (SMG), an area adjacent to AG, which is also associated with retrieval of verbally coded information; and IPS, which is associated with magnitude processing. Thus, over development, children transitioned from strategic processing to a mixture of symbol processing, automatic (verbal) memory retrieval, and magnitude processing. This qualitative shift, which was opaque given the behavioral data alone, highlights how neuroscience data can provide unique insights into mathematical development.

Embodied Understanding of Numbers and Arithmetic

In what sense are numbers and arithmetic meaningful? A conventional answer in logic and the philosophy of mathematics is that mathematical symbols and expressions denote sets of objects, and are essentially abstract even when the objects themselves are concrete. An emerging alternative in mathematical cognition is that mathematical symbols and expressions gain meaning through their grounding in the perceptual and motor systems of the body (Lakoff & Núñez, 2000). This view is consistent with the emphasis that mathematics education places on manipulatives such as Cuisenaire rods for building mathematical understanding (Montessori, 1966). It is for this reason that many curricula have children first use manipulatives to help them construct an informal semantics for new mathematical concepts before introducing them to the relevant symbolic formalism.

Although we have emphasized the abstract nature of number – numerosity is a property of a set – humans also think about number in a more embodied way. Many cultures have ways of representing numerosities and counting practices in terms of body parts. We count and show numbers on our fingers because fingers are handy sets to manipulate. These days we use our fingers for numbers up to 10, and you would be forgiven for thinking that we simply hold up as many fingers as the numerosity we wish to convey. However, it is more complicated than that. First, even in presenting numbers to 10, there are cultural conventions. For example, in Northern Europe 1 is presented as the index finger, but in Southern Europe it presented as the thumb, and in Japan as bending the little finger. Even in Europe there existed a traditional method for presenting numbers up to 10 000 on the fingers, which seems to have died out. Illiterate cultures without specialized number words use other body parts in addition to fingers to represent numbers higher than 10. The Yupno of Papua New Guinea count up to 33 using their toes, eyes, ears, nose, nostrils, nipples, belly-button, testicles, and penis (see Butterworth, 1999, Chapter 5, for more details of these practices).

We also think of numbers in a spatial way. This is partly because we see numbers spatially arrayed in everyday life: on clock faces, written out horizontally, and so on. In school, the digits 1 to 10 are almost always written from left to right, and number lines used in teaching similarly put small numbers on the left and large numbers on the right. There is even an unconscious association between small numbers and the left of space, and large numbers and the right side. This was first demonstrated by Dehaene and colleagues. They asked subjects to judge whether a number between 1 and 9 was odd or even, and press a left-hand button for odd, and a right-hand button for even (and vice versa, of course). They found that the response was faster with the left hand for small numbers, and with the right hand for large numbers. They memorably called this the "spatial–numerical association of response codes", or SNARC effect (Dehaene, Bossini, & Giraux, 1993). This effect has been replicated many times. Indeed, the neural basis for the SNARC effect involves the same areas as number processing in the parietal cortex, which is also involved in spatial cognition, suggesting an intrinsic relationship between number and space (Hubbard, Piazza, Pinel, & Dehaene, 2005; Rusconi, Turatto, & Umiltà, 2007). However, there is evidence to suggest that the mental relationship between numbers and space depends on experience and on task. So, for example, in Dehaene's original study, a group of Iranians who write from right to left showed the reverse relationship – a left-hand advantage for large numbers and right-hand advantage for small numbers. Also, if subjects are asked to study a clock face, where the larger numbers are on the left, before the task, this also reverses the SNARC effect (Bächtold, Baumüller, & Brugger, 1998). The SNARC effect seems to depend on the task requiring a lateralized motor response – such as left and right hands.

The close connection between mathematical thinking on one hand and visuomotor processing on the other – and the neural bases of this connection – have been known to neuropsychologists for years. The *Gerstmann syndrome* is a combination of four impairments, two involving symbol systems (dyscalculia and dysgraphia) and two involving the visuomotor system (left–right disorientation and finger agnosia). It is associated with lesions to left AG. Gerstmann himself thought that the key to the syndrome was an impaired "body schema," that showed up in the left–right disorientation, and particularly in finger agnosia. Rusconi, Walsh, and Butterworth (2005) demonstrated that applying rTMS over left AG disrupts performance on both a number task and a finger gnosia task. Nevertheless, the functional relationship between the neural representation of fingers and calculation has been questioned. Since both involve the left AG, it could be simply that damage that affects the neural representation of fingers, especially hand shapes that could be used to present numbers, is likely to also affect a functionally independent but anatomically neighboring calculation system (Rusconi, Pinel, Dehaene, & Kleinschmidt, 2010).

Nevertheless, there is emerging evidence of an association between visuospatial and motor skills on one hand and mathematical achievement on the other. Fayol, Barrouillet, and Marinthe (1998) found that individual differences in motor ability predict individual differences in mathematical achievement. Using a longitudinal design, measures of finger agnosia, similar to those used with neurological patients, were collected in children at age 5. These included a task in which the children's eyes were closed, and the experimenter touched one finger twice or two fingers. The child, with eyes open, then had to point to the fingers touched. Measures of general intellectual development and mathematical achievement (number and arithmetic concepts) were collected at ages 5 and 6. Finger gnosia measures were a significant predictor of mathematical achievement, even after the effects of general intellectual development and age were partialed out.

Another study was able to refine these results. Noël (2005) tested five-year-olds for finger gnosia and left–right disorientation. Fifteen months later, numerical and reading abilities were assessed. She found that performance in both the finger gnosia and the left–right test were good predictors of numerical skills one year later, but not good predictors of reading skills, which proves their specificity to mathematics. However, because the left–right test was also a predictor, Noël suspected that these tests are just picking up the development of the parietal cortex, not the functional role of fingers.

She further explored this in a training study. If finger representations do aid the development of number concepts and arithmetic, training on finger tasks should improve the acquisition of arithmetical skills in young children. Bafalluy and Noël (2008) trained children in Grade 1 who were found to have poor finger gnosia on finger representation; this improved both their finger gnosia to better-than-average levels, and also their arithmetical performance. "These results indicate that improving finger gnosia in young children is possible and that it can provide a useful support to learning mathematics" (Bafalluy & Noël, 2008).

Bodily activities play other roles in developing arithmetic. In particular, fingers are widely used by young children in the early stages acquiring addition skills. For example, Geary, Hoard, Byrd-Craven, and DeSoto (2004) found that American kindergarteners used fingers on 29% of addition trials where the sum was less than 11, and 76% of trials where it was more than 10. In first and second grade, they were still using fingers on 35% of trials. By contrast, Chinese children of the same age did not use fingers at all (Geary, Bow-Thomas, Liu, & Siegler, 1996). Geary et al. note that "The use of fingers during counting appears to be a working memory aid that allows the child to keep track of the addends physically, rather than mentally, during the process of counting."

In a study on preschoolers, pointing at objects to be counted helps coordinate counting words and objects to be counted, and helps segregate items counted from those to be counted (Alibali & DiRusso, 1999).

Individual Differences in Mathematical Achievement

Why are some people better at mathematics than others? The question of individual differences is critical for mathematics education. There are of course many causes that could affect all school subjects, such as general intellectual ability, working-memory capacity, socioeconomic status, educational experiences, missing school, conduct difficulties, self-esteem, and so on. Here we focus on differences that are specific to learning mathematics.

For example, consider the distance effect, defined as the difference in response times between slower near-distance comparisons (e.g., 8 versus 9) and faster far-distance comparisons (e.g., 1 versus 9). This effect has been replicated hundreds of times in the literature. However, not *every* individual shows a distance effect; an average, after all, is typically made up of rather unaverage people. Most people exhibit a distance effect – but some show no effect of distance, and a few even show an *inverse* distance effect, i.e., are faster for near-distance versus far-distance comparisons! Individual differences such as these are proving to be more than statistical noise. Even typically developing people differ quantitatively in their distance effects, and potentially in their neural representations of mathematical concepts. This raises the question of whether individual differences in mathematical cognition are associated with individual differences in mathematical achievement. This question has been addressed in a number of important studies that span the developmental range.

Primary and middle childhood (5–12)

Holloway and Ansari (2009) used a cross-sectional design to investigate whether individual differences in number representation are associated with individual differences in mathematical achievement. They measured the symbolic and nonsymbolic distance effects in six-, seven-, and eight-year-old children, as well as their mathematical achievement test scores. The size of a child's symbolic distance effect predicted his or her arithmetic fluency (i.e., solving one-digit arithmetic problems in a timed manner), even after controlling for age, raw processing speed, and other general variables. By contrast, the size of the nonsymbolic distance effect did not predict arithmetic fluency. These findings suggest that the fidelity of the mapping between number symbols and magnitude representations – but not necessarily the fidelity of the magnitude representations themselves – is related to mathematics achievement.

De Smedt, Verschaffel, and Ghesquière (2009) used a longitudinal design to sharpen these cross-sectional findings. They measured the symbolic distance effect of children in first grade, and their mathematical achievement in second grade. The size of a child's symbolic distance effect in first grade predicted mathematical achievement in second grade, even after controlling for fluid intelligence, processing speed, and age. The longitudinal nature of this design is stronger evidence that a connection between number symbols and magnitude representations is important for mathematical achievement.

Secondary school and adolescence (12–18)

We are aware of only one study of students in secondary school investigating whether individual differences in mathematical cognition are associated with individual differences in mathematical achievement. Halberda et al. (2008) had 14-year-olds perform a nonsymbolic comparison task. They found that individual differences in the size of the nonsymbolic distance effect were retroactively associated with individual differences in mathematical achievement in kindergarten through third grade. This association was probably not due to maturational factors: the correlation remained significant even after general intelligence, working memory, visuospatial ability, and other general ability measures in third grade were partialed out. This finding that the fidelity of one's magnitude representations may be important for mathematical achievement is inconsistent with the Holloway and Ansari (2009) findings, demonstrating the need for further research.

Lifelong learning (adulthood)

Grabner et al. (2007) investigated the neural correlates of individual differences among adults, focusing on arithmetic ability. They included two groups of participants, one low in mathematical competence and one high, who were otherwise comparable in age and general intelligence. Multiplication fluency was measured by having participants solve one-digit problems, and multiplication calculation was measured by having them solve multidigit problems. At the group level, high-competence participants showed greater activation in left AG than low-competence participants. Critically, this was also true at the individual level: mathematical competence was positively correlated with left AG activation, even after overall processing speed was partialed out. Recall that left AG has been implicated in the retrieval of arithmetic facts. Thus, this study suggests that one source of individual differences in mathematical competence is the ability to use relatively fast and effortless memory retrieval when solving arithmetic problems, as opposed to relatively slow and effortful strategic processing.

Dyscalculia

Developmental dyscalculia is usually and rather broadly defined as a low mathematical achievement in the presence of otherwise normal intelligence and access to educational resources. Current prevalence estimates are between 3% and 6% (Reigosa-Crespo et al., 2011; Shalev, 2007), which is roughly one child in every classroom.

Dyscalculia has been neglected both in research support compared with other neurodevelopmental disorders (Bishop, 2010) and in public recognition, even though its impact on life chances can be at least as damaging as, for example, dyslexia (Parsons & Bynner, 2005). Recent research has revealed that dyscalculia is a congenital condition, often inherited, that can persist into adulthood. It can occur in the presence of normal or superior intelligence and working memory (Landerl, Bevan, & Butterworth, 2004); see Butterworth et al. (2011) for a review.

Thus, it appears to be a deficit specific to learning mathematics, or more particularly to learning arithmetic.

Dyscalculia as a core deficit in processing numerosities

We have argued that dyscalculia is due to a *core deficit* in representing and processing numerosities (Butterworth, 2005, 2010; Butterworth et al., 2011). People with dyscalculia lack an intuitive sense of the numerosities of sets. This impairs their understanding of the number symbols defined in reference to these numerosities, and ultimately their understanding of arithmetic operations defined over number symbols. The result is low arithmetical achievement beginning in elementary school. This impairment has far-reaching effects, as number and arithmetic are foundational for higher-level mathematics, from algebra to calculus and beyond.

It is important from both a practical and a theoretical perspective to distinguish learners who are bad at math from those who are dyscalculic. As noted above, there are many reasons for poor math, including lack of access to appropriate education. Thus, to be dyscalculic means not just poor at math compared with peers, but to have the core deficit as well. To assess for dyscalculia therefore requires tests of very simple numerosity processing, that depend only minimally on access to appropriate education. For example, Butterworth (2010) and Butterworth et al. (2011) have argued for tests of enumerating small sets and for comparing small numbers.

This distinction is important because different studies of dyscalculia have adopted different inclusion criteria, and this has led to conflicting results. Some studies have adopted overly broad criteria, with dyscalculic groups likely included both real dyscalculics and those who were simply bad at mathematics. For example, Rousselle and Noël (2007) defined their dyscalculic group as children scoring below the 15th percentile on a standardized test of mathematical achievement.

Dyscalculia in primary and middle childhood (5–12)

The core-deficit hypothesis is consistent with the finding reviewed above that individual differences in elementary-school mathematical achievement are retroactively predicted by the size of the *nonsymbolic* distance effect in adolescence (Halberda et al., 2008; Piazza et al., 2010). Several studies have shown that simple numerosity, such as naming the number of objects in a set or comparing the numerosity of two sets, is defective in dyscalculia (see Butterworth, 2010, for a review).

Another possibility is that dyscalculia is a deficit in *mapping* number symbols onto intact representations of numerical magnitude. This mapping hypothesis is consistent with the findings reviewed above that mathematical achievement in elementary school is predicted by the size of the *symbolic* distance effect (Holloway & Ansari, 2009; De Smedt et al., 2009). Thus, the predictive value of the nonsymbolic versus symbolic distance effect is important for distinguishing between the numerosity and mapping types of deficit. The mapping hypothesis was supported by early studies in the literature, which we review first. However, later studies, which have adopted better inclusion criteria, find evidence for poor discrete magnitude representations – and thus the numerosity hypothesis – at both the behavioral and brain levels.

Rousselle and Noël (2007) compared a group of typically developing second graders and a group with dyscalculia. The dyscalculic group was slower than the control group for symbolic comparison after controlling for age and overall processing speed. By contrast, the two groups were comparable for nonsymbolic comparison. This study supports the hypothesis of a mapping deficit. However, because a rather liberal inclusion criterion for the dyscalculia group was used – mathematical achievement 1.5 standard deviations below the mean – it is possible that these results were driven by typically developing children at the low end of the normal curve rather than by true dyscalculics. More generally, studies that use a liberal inclusion criterion often find conflicting results. Taking just the bottom 5% s dyscalculic, a study of 8- to 10-year-old children found that the dyscalculic group was slower than the control group on both symbolic and nonsymbolic comparison (Landerl & Kölle, 2009).

We can turn to the neuroscience data to more accurately identify the core deficit in dyscalculia. Kucian et al. (2006) conducted the first fMRI study of an neuropathology underlying dyscalculia. A group of 11-year-old children with dyscalculia and a group of age-matched controls performed a nonsymbolic comparison task and an approximate addition task, both of which likely tapped discrete magnitude representations. The largest group differences were observed on an approximate addition task, where the dyscalculic group displayed less activation in a bilateral frontoparietal network including MFG, IFG, and IPS. The finding of a neural difference for an approximate reasoning task supports the hypothesis that the numerosity representations themselves be defective, not just the mapping from symbol to the representations. An additional insight from this study is that there is no qualitative difference between the arithmetic networks of typically developing children and those with dyscalculia – both activated the same network of frontal and parietal area. Rather, there is a quantitative difference in the degree to which the network is activated. In particular, the lower IPS activation in the dyscalculic group implicates a weakened representation of numerosity.

Price et al. (2007) investigated differences between a group of 12-year-old children with dyscalculia and a group of age-matched controls using a nonsymbolic comparison task. (Dyscalculia was defined liberally, as mathematical achievement at least 1.5 standard deviations below the mean.) Behaviorally, the two groups were comparable. However, the fMRI data revealed an important cortical difference: whereas the control group showed the typical neural distance effect in right IPS, the dyscalculia group did not. These results suggest a weakened magnitude representation in people with dyscalculia, consistent with the numerosity proposal. They also demonstrate how neuroscience methods can reveal differences not visible at the behavioral level.

There are also structural differences in the brains of dyscalculics that turn out to be in the left and right IPS, precisely the brain areas implicated in numerosity processing (see Figure 8.2). The first such study was by Isaacs, Edmonds, Lucas,

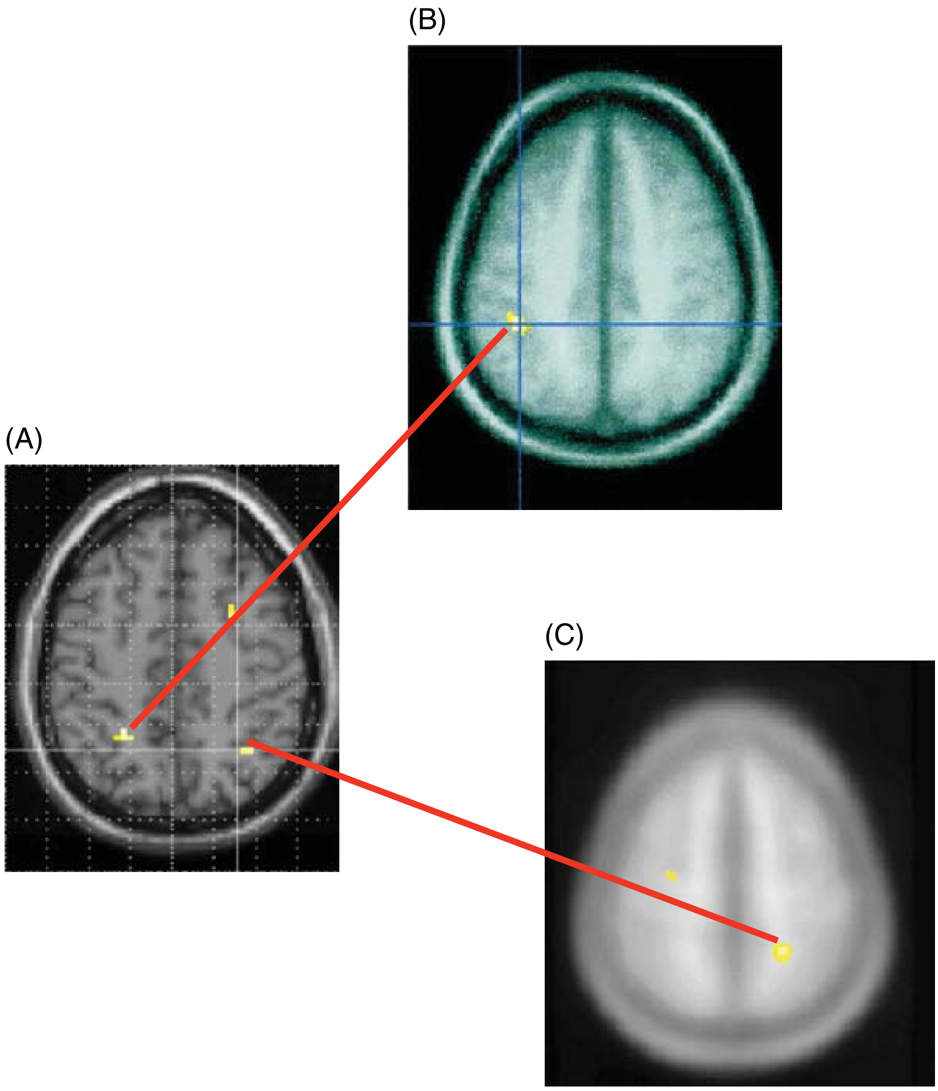


Figure 8.2 Reduced grey-matter density in the numerosity-processing regions of dyscalculic brains. (A) Brain areas dedicated to numerosity processing (Castelli et al., 2006, with permission). (B) Reduced grey-matter density in left parietal numerosity-processing area in dyscalculic adolescents (Isaacs et al., 2001, with permission). (C) Reduced grey-matter density in right parietal numerosity-processing area in dyscalculic children (Rotzer et al., 2008, with permission).

and Gadian (2001), who found a reduction in grey-matter density in a small region in the left IPS; Rotzer et al. (2008) subsequently found reduced grey matter density in the right IPS in younger subjects, reflecting the developmental trajectory mentioned above, that simple number processing, especially of non-symbolic numerosities, goes from a predominantly right hemisphere locus to a bilateral one.

Lifelong learning

Developmental dyscalculia can persist into adulthood, though it is unclear what proportion of early dyscalculic learners remains in this condition. One study by Shalev, Manor, and Gross-Tsur (2005) found that some 40% of 11-year-olds were still dyscalculic at age 17, and almost all were in the bottom quartile on a standardized arithmetic task. However, Shalev et al. used a criterion of two years below age norms, but did not test specifically for a core deficit. Certainly, it is not hard to find examples of adults with severe math difficulties and a core deficit, such as "Charles," described by Butterworth (1999). There have been few published functional studies of the brains of adult developmental dyscalculics. One example that used EEG with two-digit addition problems found that older subjects tended to use both hemispheres while younger ones used the left hemisphere predominantly, and suggested that the younger subjects showed more strategic flexibility in how they solved the problems (El Yagoubi, Lemaire, & Besson, 2005). Nevertheless, thorough investigations of the effects of normal aging on the brain systems for mathematics are urgently needed, along with studies of the effects of clinical conditions that are associated with aging, such as mild cognitive impairment and the dementias.

Educating the Mathematical Brain

One obstacle for bridging from neuroscience to education is that neuroscience methods are not easily portable to the classroom. However, neuroscientists are beginning to isolate elements of mathematical thinking and learning that occur in school, and study them in carefully controlled laboratory settings. The results are pointing the way to new instruction for typical classrooms, and to promising instructional interventions for dyscalculia. It is true that, to date, neuroscience has not had the impact on education – for good or ill – that cognitive and developmental psychology has had. Thorndike's (1922) book, *The Psychology of Arithmetic*, created a focus on drilling simple number bonds. In the 1930s, Brownell, in several important publications, applied psychological ideas about meaningful practice to how math should be taught (Brownell, 1928, 1935, 1938).

In the 1950s and 1960s, Piaget's "constructivist" theories about the nature of cognitive development were very influential. Constructivism emphasizes the child's construction of new schemas (accommodation) when new stimuli cannot be understood using existing schemas (assimilation). Both Montessori in her schools, and Gattegno in the use of Cuisenaire rods, were deeply influenced by Piaget. Many mathematics education researchers continue to advocate constructivist instruction over direct instruction (Bransford, Franks, Vye, & Sherwood, 1989; Duffy & Jonassen, 1992; von Glasersfeld, 1989). Whereas direct instruction assumes that knowledge can be rather directly communicated ("transmitted") from teachers to students, constructivist instruction proposes that students must construct their own knowledge if it is to be meaningful. The emphasis is on particular kinds of activity such as game playing and other hands-on learning.

More recently, Johnson, Karmiloff-Smith, and Mareschal, and their colleagues, have proposed a line of research, based on Piagetian ideas, that they call "neuroconstructivism" (see Westermann, Thomas, & Karmiloff-Smith, 2010, for a recent review). In this, neural specializations in the brain are not directly inherited – the brain does not start out modularized for specific cognitive functions – but rather brain organization is shaped by interaction with the environment, and specializations emerge in a consistent way largely as a result of the common structures of experience and some intrinsic biases in neural receptivity to particular types of information.

Here we focus on two areas where neuroscience can inform education: first, methods of instruction; and, second, individual differences, including new approaches to remediating dyscalculia.

Methods of instruction

Practice and transfer Direct instruction is the term for a range of traditional classroom activities, including lecture, recitation, reviews, seatwork, homework, quizzes, and exams (see, e.g., Rosenshine, 1995). In an important early study, Delazer and colleagues investigated the neural consequences of one aspect of direct instruction, practice (Delazer et al., 2005). Adults practiced solving 18 complex multiplication problems, each involving a one-digit operand and a two-digit operand. These arithmetic facts are not normally memorized in school. Participants subsequently verified complex multiplication problems in the scanner, half of which were trained and half of which were untrained. Participants were faster and more accurate on trained versus untrained problems. More importantly, the two classes of problems activated different cortical networks. The network activated by untrained problems included bilateral IPS (associated with magnitude processing) and bilateral IFG (associated with executive function and verbal working memory). This is essentially the same frontoparietal network that Grabner et al. (2009) found when participants self-reported using effortful, strategic processing to solve simple arithmetic problems. By contrast, the trained network included left AG (associated with memory retrieval when solving simple arithmetic problems). These findings demonstrate that following practice – a component of direct instruction – there is a shift from strategy-based processing to memory-based processing, specifically retrieval of verbally/symbolically coded arithmetic facts.

Ischebeck et al. (2006) investigated whether there are different practice effects for multiplication and subtraction. Adults practiced complex multiplication and subtraction problems outside the scanner, and then verified trained and untrained problems in the scanner. Participants were again faster and more accurate on trained versus untrained problems, regardless of the operation, replicating and extending the work of Delazer et al. (2005). At the neural level, the results for multiplication also replicated the work of Delazer et al. (2005), with a training shift from a frontoparietal network (including IPS) to left AG, again suggesting a shift from strategy- to memory-based processing. Critically, the results for subtraction were different: both trained and untrained problems recruited the frontoparietal network associated with strategy-based processing. This suggests that the neural consequences of practice are contingent on the mathematical concept being practiced.

To sum up the results of these two studies, for complex multiplication, the improvement in behavioral performance following practice is a function of a shift from strategic to memory-based processing. This is the analogous to the behavioral patterns and cortical shift that Rivera et al. (2005) documented for simple addition and subtraction over development, where the left inferior parietal lobe becomes increasingly specialized for addition. By contrast, in the case of complex subtraction, the continuous improvement in behavioral performance following practice is a function of increased efficiency in the frontoparietal network associated with strategic processing (Ischebeck et al., 2006). These findings are important, suggesting that if the pedagogical goal is to automatize subtraction, then mathematics educators should look beyond direct instruction methods.

Individual differences

Neuroscience can help identify cognitive strengths and weakness in individual learners in a way that can inform the design of an educational context appropriate for that learner. This means ensuring that the learning context is adaptive to the learner's current needs and "zone of proximal development" (Vygotsky, 1978). This will include specifying the content that needs to acquired and a pace of progression suitable for the learner.

Dyscalculia This approach is perhaps best exemplified in the learning contexts designed for dyscalculic learners. We have suggested that neuroscience has identified the core deficit, that is, the target for intervention – a deficit in processing numerosities. This does not however specify how dyscalculic learners can be helped.

One needs to turn to *pedagogic principles* and the best practice of special educational needs (SEN) teachers to design appropriate instruction. From a pedagogical perspective, activities that require the manipulation of concrete objects provide tasks that make number concepts meaningful by providing an intrinsic relationship between a *goal*, the *learner's action*, and the *informational feedback* on the action. This kind of feedback provides intrinsic motivation in a task, and this is of greater value to the learner than the extrinsic motives and rewards provided by a supervising teacher (Bruner, 1961; Deci, Koestner, & Ryan, 2001).

Experienced SEN teachers will use Cuisenaire rods, number tracks, and playing cards to give learners experience of the meaning of number. Through playing games with these physical objects, learners can discover from their manipulations, for example, which rod fits with an 8-rod to match a 10-rod, or how many beads to put out on the track to get from the given number to desired number, and so on (Butterworth & Yeo, 2004). These tasks afford discovery learning and the *construction* of solutions, which in turn enable learners to compare their solution with the correct solution, and if necessary adjust their own solution. These are powerful mechanisms for learning with understanding (Papert, 1980; Piaget, 1952).

Ideally, the experienced teacher will adapt the activities to match the learner's current level of understanding, and find ways to push the understanding into the zone of proximal development. However, this may require extensive one-to-one teaching, which may not always be possible, and which, in any case, will be expensive.

Tasks adapted to the learner's current level can now be achieved using software games that embody the pedagogic principles outlined above. The *Number Race* and *Graphogame-Maths* are adaptive games based on neuroscience that target basic numerosity processing, and appear to be effective (Räsänen, Salminen, Wilson, Aunioa, & Dehaene, 2009). However, neither requires learners to manipulate numerical quantities. Manipulation is critical for providing an intrinsic relationship between a task goal, the learner's action, and informational feedback on that action.

An approach that emulates the manipulative tasks used by SEN teachers has been taken in adaptive software that enables the construction of a solution, provides informational feedback, and offers a means to match the learner's solution to the correct solution in the case of error. See Butterworth et al. (2011) for further discussion. Examples can be downloaded from http://www.numbersense.co.uk/.

These games have not been subjected to large-scale evaluation, but one important advantage of adaptive software is that learners can do more practice per unit time than with a teacher. Thus, 12-year-old SEN learners using a number bonds game managed 4–11 trials per minute, while in an SEN class of three supervised learners only 1.4 trials per minute were completed during a 10 min observation. In another SEN group of 11-year-olds, the game elicited on average 173 learner manipulations in 13 min (where a perfect performance, in which every answer is correct, is 88 in 5 min, since the software adapts the timing according to the response) (Butterworth & Laurillard, 2010).

Butterworth et al. (2011) conclude that "At present it is not yet clear whether early and appropriately-targeted intervention can turn a dyscalculic into a typical calculator. Dyscalculia may be like dyslexia in that early intervention can improve practical effectiveness without making the cognitive processing like those of the typically developing."

In dyslexia research, appropriate phonological training can have the effect of making patterns of neural activity more like those in typical readers (Eden et al., 2004). This is important, since it takes the measurement of the effects of an intervention beyond behavior into its underlying mechanisms. Is the same true for dyscalculia? So far, there has only been one study published about the effects of this kind of intervention on patterns of neural activity. In this study, by von Aster's group in Zurich, nine-year-old typical learners and matched nine-year-old dyscalculics (1.5 SD below average) were trained using a specially designed computer game (Kucian et al., 2011). The game required landing a spaceship on a number line from 0 to 100, according to the number on the spaceship, or simple calculation on the spaceship (see Figure 8.3(A)). The game was played for 15 min a day, 5 days a week, for 5 weeks. The effects of the training were assessed behaviorally, and were effective for both dyscalculic and typical learners, with a bigger effect for the dyscalculics, who nevertheless remained worse than the controls (see Figure 8.3(B)). Activation was measured in an fMRI task that required the child to determine whether three numbers were in ascending or descending order, compared with a control task in which they had to determine whether the digit "2" was present. In this situation, dyscalculics showed less parietal activation, but more frontal activation. The authors conclude that the "results lend further support to a deficient number

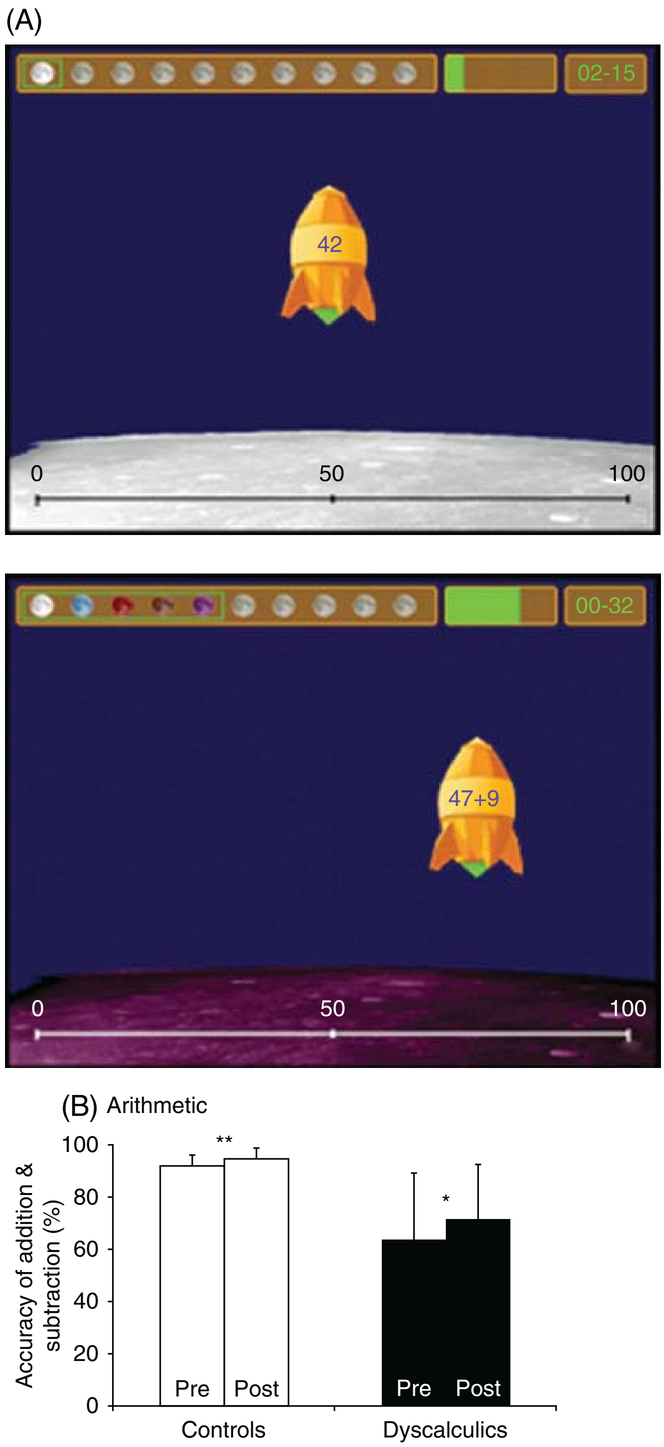


Figure 8.3 Effects of training on arithmetical performance and patterns of neural activity. Nine-year-old dyscalculic and typical learners were trained on a simple arithmetical tasks using a computer game for 15 min a day, 5 days a week, for 5 weeks. (A) The game "Rescue Calcularis" required the learner to land the spaceship on the number line below. In the top panel, the task was to land it at 42; in the bottom panel, it was to land at 18, the solution to 27 – 5. (B) Training was effective for both groups, but more for the dyscalculics (black bars). Nevertheless, they still failed to reach typical levels of performance after the training.

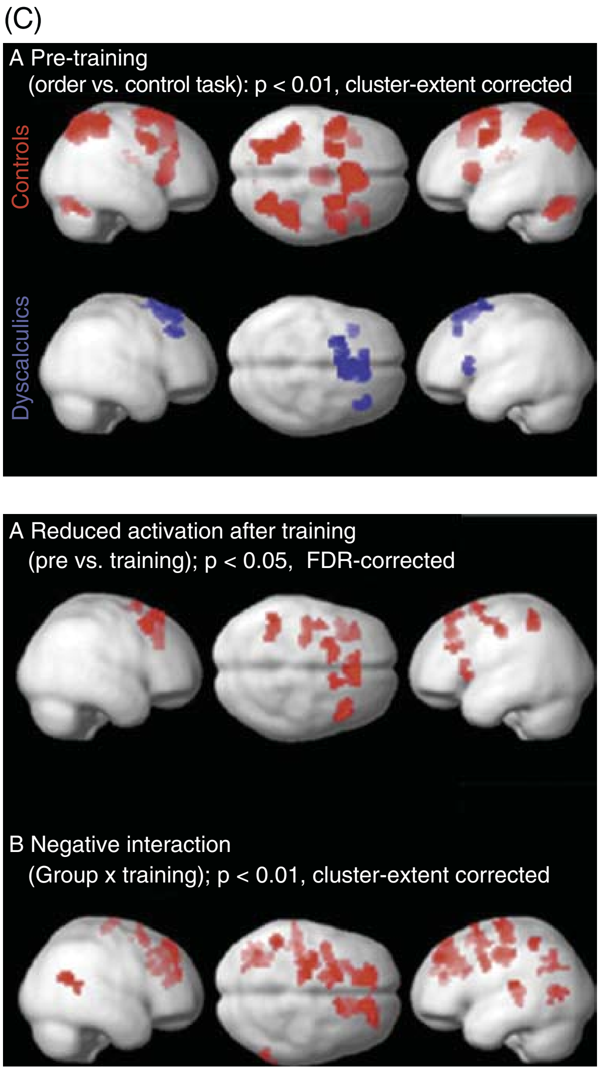


Figure 8.3 (*continued*) (C) Top panel: group differences in brain activity in a different number task before training (see text). The dyscalculics showed more frontal activation in the task compared with a control task. Bottom panel: training reduced frontal activity in both groups, but more so in the dyscalculics (negative interaction). (From Kucian et al. (2011) with permission.)

representation in the parietal lobe associated with dyscalculia, causing stronger engagement of supporting frontal lobe functions such as working memory and attentional control to solve a numerical task" (p. 792). The effect of training was striking. In both groups, there was a reduction in frontal activation, suggesting that the training transferred to the fMRI task, making it more automatic and thus dependent on parietal areas, and less strategic and thus dependent on frontal areas. This effect was even more marked in the dyscalculics. The effects of training, therefore, tended to move the dyscalculics to a more typical pattern of both behavior and neural activity, paralleling the shift observed in dyslexia training studies.

Future Directions

Although neuroscience studies of mathematical thinking are in their infancy, they are already shedding light on topics of great relevance to education, including mathematical development, the spatial basis of mathematical concepts, the nature of individual differences in achievement, the neural correlates of different instructional approaches, the core deficit in dyscalculia, and the design and evaluation of effective remediation. Mathematical thinking is already an important bridge between education and neuroscience, and its importance will only grow. A scientific explanation of the neural bases of mathematics is necessary for an evidence-based education: for understanding why some instructional interventions (but not others) work for some children (but not others), and for informing the design of new instruction. Conversely, mathematics is one of the core symbol systems of human culture, and investigating the neural bases of this symbol system cannot help but generate new empirical paradigms and theoretical explanations that will enrich our understanding of the brain more generally.

Much of the research to date has focused on natural numbers and arithmetic operations defined over them. Comparatively little is known about the psychological and neuroscience underpinnings of more abstract and advanced concepts in mathematics such as negative numbers, place value, and algebra. We briefly review some initial attempts to fill this gap here. We also preview emerging research on the benefits of neural stimulation for mathematics learning.

Negative numbers

Recent research has illuminated how adults mentally represent negative numbers. Adults show a symbolic distance effect for comparisons of negative numbers that parallels the one for comparisons of natural numbers, for example comparing −1 versus −4 is faster than −1 versus −-9. Some interpret this as evidence that negative numbers are mentally represented as magnitudes (Varma & Schwartz, 2011). Others argue that negative numbers do not have magnitude representations, but are instead mapped to natural numbers via symbolic rules (Tzelgov, Ganor-Stern, & Maymon-Schreiber, 2009). Under this account, when comparing negative numbers (e.g., which of −1 versus −4 is greater?), people first strip the negative signs, then reverse the judgment (e.g., which of 1 versus 4 is lesser?), and finally consult magnitude representations of natural numbers.

Mixed comparisons of negative numbers and natural numbers have the potential to resolve this debate. Tzelgov et al. (2009) found no effect of distance for mixed comparisons, with near comparisons (e.g., −1 versus 2) made as fast as far comparisons (e.g., −1 versus 7). They interpreted this as evidence that people use symbolic rules such as "positives are greater than negatives." However, more recent studies have found distance effects for mixed comparisons (Gullick, Wolford, & Temple, 2011; Krajcsi & Igács, 2010; Varma & Schwartz, 2011). Surprisingly, these effects have been in the inverse direction, with near comparisons faster than far comparisons, suggesting that the magnitude representations of negative numbers are spatially transformed.

Neuroscience data can potentially inform whether negative numbers are understood as magnitudes or using rules. Early studies are finding neural distance effects in bilateral IPS for comparisons of negative numbers and for mixed comparisons of negative numbers and natural numbers, consistent with magnitude representations (Blair, Rosenberg-Lee, Tsang, Schwartz, & Menon, 2010; Gullick et al., 2011). However, these studies also find neural distance effects in prefrontal areas associated with controlled rule processing. Further research is required to untangle the neural correlates of negative number understanding.

Some researchers are beginning to study the cognitive development of integer understanding (Varma & Schwartz, 2011). Others are asking the educational neuroscience question of whether different kinds of instruction – focusing on symbolic rules such as "positive particles cancel negative particles" versus visuomotor movements along number lines – set up different kinds of representation, with correspondingly different neural correlates (Tsang, Blair, Bofferding, Rosenberg-Lee, & Schwartz, 2011). Again, we expect these to be fruitful areas for future research.

Place value

Place-value notation is a generative system for naming numbers using a small set of symbols. Our base-10 system uses the number symbols 0–9 plus a few extra symbols (" . ", "–") to name very large and very small numbers using relatively few digits. Place-value notation is important because its structure supports the standard algorithms for "long" arithmetic, for example enabling "borrowing" and "carrying." For this reason, mastering place-value notation is an important goal of early elementary education

Early research on place-value notation focused on how adults and children understand multidigit natural numbers. Consider the task of judging which of 79 versus 17 is greater. Initial studies suggested that, for large numbers such as these, people do not directly consult magnitude representations. Rather, they understand them as composite representations, sequentially comparing the face value of each place from left to right until a judgment is possible (Hinrichs, Berie, & Mosell, 1982; Hinrichs, Yurko, & Hu, 1981). Although some studies have challenged this finding (Dehaene, Dupoux, & Mehler, 1990), the results have largely held up. A particularly diagnostic finding is the *incompatibility effect*: when comparing two-digit numbers, response times are slower when the judgment based on the tens places conflicts with the judgment based on the ones place. For example, people are slower to compare 81 versus 19 than 79 versus 17 (Nuerk, Weger, & Willmes, 2001). Developmental studies of the incompatibility effect indicate that children have the adult composite representation of multidigit numbers as early as second grade (Landerl & Kölle, 2009; Nuerk, Kaufmann, Zoppoth, & Willmes, 2004).

Neuroscientists are identifying the neural correlates of place value in adults. Administering TMS over left AG while adults compare two-digit numbers disrupts number comparison (Göbel, Walsh, & Rushworth, 2001). Recall that this area is associated with the retrieval of symbolically or verbally coded arithmetic facts, and *not* with magnitude processing. This finding suggests that the composite representation of multidigit numbers is accessed and processed using symbolic rules. Further evidence was provided by a study investigating the neural basis of the incompatibility effect, which found greater activation for incompatible versus compatible comparisons in left SMG, an area adjacent to left AG (Liu, Wang, Corbly, Zhang, & Joseph, 2006). However, this study also found neural incompatibility effects in a number of other areas, including bilateral occipitotemporal cortex associated with processing the visual forms of numbers, prefrontal areas associated with controlled symbolic and attentional processing, and IPS, which is associated with magnitude processing. Thus, the neural representation of multidigit numbers remains an open question.

The emerging scientific understanding of how adults and children understand place value promises to inform progress in education. For example, a recent study found larger incompatibility effects in elementary-school-aged children with dyscalculia versus those who were typically developing, indicating that weakened composite representations of very large numbers is associated with low math achievement (Landerl & Kölle, 2009). Further research following up on this tantalizing result is needed.

Algebra

Anderson and colleagues have conducted a series of fMRI studies of algebra problem solving. These studies have been driven by a theoretical model that assigns to left IFG the function of retrieving information from long-term memory, and to left posterior parietal cortex (PPC; an area posterior to IPS and AG) the function of maintaining and transforming mental representations (Anderson, 2007). These functions are critical for solving simple algebra equations (e.g., $x/3 + 2 = 8$), which requires both retrieving relevant arithmetic facts (e.g., $8 - 2 = 6$) and applying these facts to transform the current equation into a newer, simpler equation (e.g., $x/3 + 2 = 8 \rightarrow x/3 = 6$). As predicted by the model, the greater the number of arithmetic facts that must be retrieved to solve an algebra equation, the greater IFG activation, and the greater the number of transformations that must be performed, the greater PPC activation (Danker & Anderson, 2007; Stocco & Anderson, 2007).

Building on this basic finding, Sohn et al. (2004) investigated the neural processes associated with solving algebra equations versus story problems. They found increased bilateral PPC activation when solving algebra equations, consistent with the sequential transformations. By contrast, they found greater left IFG when solving story problems, consistent with this area's role in verbal working memory and "expressive" language.

Qin et al. (2004) took the next step towards educational relevance, investigating the effect of practice in a sample of children aged 12–15. The children practiced solving multistep algebra equations over the course of 5 days. Behavioral performance improved, of course. More interestingly, both left IFG and left PPC were less active after training, particularly as the number of transformations a problem required increased. This pattern was different from that observed in an earlier study of adults, who only showed a practice effect in left IFG (Qin et al., 2003). Taken together, these studies suggest that practice will have different effects in children versus adults.

The first neuroscience studies of fractions (Schmithorst & Brown, 2004), calculus (Krueger et al., 2008), and other advanced mathematical topics are beginning to appear, although much work remains to be done. Educational neuroscientists are also taking the first tentative steps towards incorporating neural measures of mathematical understanding into computer tutors, which typically depend solely on behavioral measures such as number of problems correct (Anderson, Betts, Ferris, & Fincham, 2010, 2012). We expect rapid progress in understanding the neural bases of abstract mathematical thinking and applying these insights to mathematics education in the future.

Neural stimulation

One study has looked at the effect transcranial direct current stimulation (TDCS) on learning novel symbols (Cohen Kadosh, Soskic, Iuculano, Kanai, & Walsh, 2010). During TDCS, a weak current is applied constantly over time to enhance (anodal stimulation) or reduce (cathodal stimulation) the excitation of neuronal populations, with maximal effect on the stimulated area beneath the electrodes. Anodal stimulation over the right parietal improved learning of the novel symbols designed to be equivalent to numbers, and this improvement lasted until retesting six months later. This study suggests that more direct intervention in neural processes could help learning, especially for those struggling such as dyscalculics.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development*, *14*, 37–56.

Anderson, J. R. (2007). *How can the human mind occur in the physical universe?* Oxford: Oxford University Press.

Anderson, J. R., Betts, S. A., Ferris, J. L., & Fincham, J. M. (2010). Neural imaging to track mental states while using an intelligent tutoring system. *Proceedings of the National Academy of Sciences of the United States of America*, *107*, 7018–7023.

Anderson, J. R., Betts, S. A., Ferris, J. L., & Fincham, J. M. (2012). Tracking children's mental states while solving algebra equations. *Human Brain Mapping*, *33*, 2650–2665.

Ansari, D., & Dhital, B. (2006). Age-related changes in the activation of the intraparietal sulcus during nonsymbolic magnitude processing: An event-related functional magnetic resonance imaging study. *Journal of Cognitive Neuroscience*, *18*, 1820–1828.

Ansari, D., Garcia, N., Lucas, E., Hamon, K., & Dhital, B. (2005). Neural correlates of symbolic number processing in children and adults. *NeuroReport*, *16*, 1769–1773.

Antell, S. E., & Keating, D. P. (1983). Perception of numerical invariance in neonates. *Child Development*, *54*, 695–701.

Bächtold, D., Baumüller, M., & Brugger, P. (1998). Stimulus–response compatibility in representational space. *Neuropsychologia*, *36*, 731–735.

Bafalluy, M. G., & Noël, M.-P. (2008). Does finger training increase young children's numerical performance? *Cortex*, *44*, 368–375. DOI 10.1016/j.cortex.2007.08.020

Bishop, D. V. M. (2010). Which neurodevelopmental disorders get researched and why? *PLoS ONE*, *5*(11), e15112. DOI: 10.1371/journal.pone.0015112

Blair, K. P., Rosenberg-Lee, M., Tsang, J., Schwartz, D. L., & Menon, V. (2010, June). *Representations and rules in negative number processing*. Poster presented at the 16th Annual Meeting of the Organization for Human Brain Mapping, Barcelona, Spain.

Bransford, J. D., Franks, J. J., Vye, N. J., & Sherwood, R. D. (1989). New approaches to instruction: Because wisdom can't be told. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (pp. 470–497). Cambridge: Cambridge University Press.

Brownell, W. A. (1928). *The development of children's number ideas in the primary grades*. Chicago, IL: University of Chicago.

Brownell, W. A. (1935). Psychological considerations in the learning and the teaching of arithmetic. *The teaching of arithmetic. The tenth yearbook of the National Council of Teachers of Mathematics*. New York: Teachers College, Columbia University.

Brownell, W. A. (1938). Readiness and the arithmetic curriculum. *The Elementary School Journal*, *38*(5), 344–354.

Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review*, *31*, 21–32.

Buckley, P. B., & Gillman, C. B. (1974). Comparisons of digits and dot patterns. *Journal of Experimental Psychology*, *103*, 1131–1136.

Butterworth, B. (1999). *The mathematical brain*. London: Macmillan.
Butterworth, B. (2005). Developmental dyscalculia. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 455–467). Hove, UK: Psychology.
Butterworth, B. (2010). Foundational numerical capacities and the origins of dyscalculia. *Trends in Cognitive Sciences*, *14*, 534–541. DOI: 10.1016/j.tics.2010.09.007
Butterworth, B., & Laurillard, D. (2010). Low numeracy and dyscalculia: Identification and intervention. *ZDM Mathematics Education*, *42*, 527–539. DOI: 10.1007/s11858-010-0267-4
Butterworth, B., Varma, S., & Laurillard, D. (2011). Dyscalculia: From brain to education. *Science*, *332*, 1049–1053. DOI: 10.1126/science.1201536
Butterworth, B., & Yeo, D. (2004). *Dyscalculia guidance*. London: nferNelson.
Cantlon, J. F., Brannon, E. M., Carter, E. J., & Pelphrey, K. A. (2006). Functional imaging of numerical processing in adults and 4-y-old children. *Public Library of Science Biology*, *4*(5), e125. DOI: 10.1371/journal.pbio.0040125
Castelli, F., Glaser, D. E., & Butterworth, B. (2006). Discrete and analogue quantity processing in the parietal lobe: A functional MRI study. *Proceedings of the National Academy of Sciences of the United States of America*, *103*(12), 4693–4698.
Cockcroft, W. H. (1982). *Mathematics counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the chairmanship of Dr W H Cockcroft*. London: HMSO.
Cohen Kadosh, R., Soskic, S., Iuculano, T., Kanai, R., & Walsh, V. (2010). Modulating neuronal activity produces specific and long-lasting changes in numerical competence. *Current Biology*, *20*, 2016–2020. DOI: 10.1016/j.cub.2010.10.007
Danker, J. F., & Anderson, J. R. (2007). The roles of prefrontal and posterior parietal cortex in algebra problem-solving: A case of using cognitive modeling to inform neuroimaging data. *NeuroImage*, *35*, 1365–1377.
De Smedt, B., Verschaffel, L., & Ghesquière, P. (2009). The predictive value of numerical magnitude comparison for individual differences in mathematics achievement. *Journal of Experimental Child Psychology*, *103*(4), 469–479.
Deci, E. L., Koestner, R., & Ryan, R. M. (2001). Extrinsic rewards and intrinsic motivation in education: Reconsidered once again. *Review of Educational Research*, *71*(1), 1–27. DOI: 10.3102/00346543071001001
Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and numerical magnitude. *Journal of Experimental Psychology: General*, *122*, 371–396.
Dehaene, S., Dupoux, E., & Mehler, J. (1990). Is numerical comparison digital? Analogical and symbolic effects in two-digit number comparison. *Journal of Experimental Psychology: Human Perception and Performance*, *16*, 626–641.
Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*, 487–506.
Delazer, M., & Butterworth, B. (1997). A dissociation of number meanings. *Cognitive Neuropsychology*, *14*, 613–636.
Delazer, M., Ischebeck, A., Domahs, F., Zamarian, L., Koppelstaetter, F., Siedentopf, C. M., Kaufmann, L., & Benke, T. (2005). Learning by strategies and learning by drill – evidence from an fMRI study. *NeuroImage*, *25*, 838–849.

Duffy, T. M., & Jonassen, D. (Eds.). (1992). *Constructivism and the technology of instruction: A conversation*. Englewood Cliffs, NJ: Erlbaum.

Eden, G., Jones, K., Cappell, K., Gareau, L., Wood, F., Zeffiro, T., Dietz, N., Agnew, J., & Flowers, D. (2004). Neural changes following remediation in adult developmental dyslexia. *Neuron*, *44*, 411-422.

El Yagoubi, R., Lemaire, P., & Besson, M. (2005). Effects of aging on arithmetic problem-solving: An event-related brain potential study. *Journal of Cognitive Neuroscience*, *17*(1), 37–50. DOI: 10.1162/0898929052880084

Fayol, M., Barrouillet, P., & Marinthe, C. (1998). Predicting arithmetical achievement from neuro-psychological performance: A longitudinal study. *Cognition*, *68*, 63–70.

Fias, W., Lammertyn, J., Reynvoet, B., Dupont, P., & Orban, G. A. (2003). Parietal representation of symbolic and nonsymbolic magnitude. *Journal of Cognitive Neuroscience*, *15*, 47–56.

Geary, D. C., Bow-Thomas, C. C., Liu, F., & Siegler, R. S. (1996). Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development*, *67*(5), 2022–2044.

Geary, D. C., Hoard, M. K., Byrd-Craven, J., & DeSoto, M. C. (2004). Strategy choices in simple and complex addition: Contributions of working memory and counting knowledge for children with mathematical disability. *Journal of Experimental Child Psychology*, *88*(2), 121–151. DOI: 10.1016/j.jecp.2004.03.002

Göbel, S., Walsh, V., & Rushworth, F. S. (2001). The mental number line and the human angular gyrus. *NeuroImage*, *14*, 1278–1289.

Grabner, R. H., Ansari, D., Koschutnig, K., Reishofer, G., Ebner, F., & Neuper, C. (2009). To retrieve or to calculate? Left angular gyrus mediates the retrieval of arithmetic facts during problem solving. *Neuropsychologia*, *47*(2), 604–608.

Grabner, R. H., Ansari, D., Reishofer, G., Stern, E., Ebner, F., & Neuper, C. (2007). Individual differences in mathematical competence predict parietal brain activation during calculation. *NeuroImage*, *38*, 346-356.

Gross, J., Hudson, C., & Price, D. (2009). *The long term costs of numeracy difficulties*. London: Every Child a Chance Trust, KPMG.

Gullick, M. M., Wolford, G., & Temple, E. (2011, April) *Understanding less than nothing: An fMRI study of negative number comparisons.* Poster session presented at the annual meeting of the Cognitive Neuroscience Society, San Francisco, CA.

Halberda, J., Mazzocco, M. M. M., & Feigenson, L. (2008). Individual differences in non-verbal number acuity correlate with maths achievement. *Nature*, *455*, 665–668. DOI: 10.1038/nature07246

Hinrichs, J. V., Berie, J. L., & Mosell, M. K. (1982). Place information in multidigit number comparison. *Memory and Cognition*, *10*, 487–495.

Hinrichs, J. V., Yurko, D. S., & Hu, J. M. (1981). Two-digit number comparison: Use of place information. *Journal of Experimental Psychology: Human Perception and Performance*, *7*, 890–901.

Holloway, I. D., & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's mathematics achievement. *Journal of Experimental Child Psychology*, *103*(1), 17–29.

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience*, *6*(6), 435–448.

Isaacs, E. B., Edmonds, C. J., Lucas, A., & Gadian, D. G. (2001). Calculation difficulties in children of very low birthweight: A neural correlate. *Brain*, *124*, 1701–1707.

Ischebeck, A., Zamarian, L., Siedentopf, C., Koppelstatter, F., Benke, T., Felber, S., & Delazer, M. (2006). How specifically do we learn? Imaging the learning of multiplication and subtraction. *NeuroImage*, *30*(4), 1365–1375.

Iuculano, T., Tang, J., Hall, C. W. B., & Butterworth, B. (2008). Core information processing deficits in developmental dyscalculia and low numeracy. *Developmental Science*, *11*(5), 669–680.

Izard, V., Dehaene-Lambertz, G., & Dehaene, S. (2008). Distinct cerebral pathways for object identity and number in human infants. *PLoS Biology*, *6*, e11. DOI: 10.1371/journal.pbio.0060011

Koshmider, J. W., & Ashcraft, M. H. (1991). The development of children's mental multiplication skills. *Journal of Experimental Child Psychology*, *51*, 53–89.

Krajcsi, A., & Igács, J. (2010). Processing negative numbers by transforming negatives to positive range and by sign shortcut. *European Journal of Cognitive Psychology*, *22*, 1021–1038.

Krueger, F., Spampinato, M. V., Pardini, M., Pajevic, S., Wood, J. N., Weiss, G. H., Landgraf, S., & Grafman, J. (2008). Integral calculus problem solving: An fMRI investigation. *Neuroreport*, *19*, 1095–1099.

Kucian, K., Grond, U., Rotzer, S., Henzi, B., Schönmann, C., Plangger, F., Gälli, M., Martin, E., & von Aster, M. (2011). Mental number line training in children with developmental dyscalculia. *NeuroImage*, *57*(3), 782–795.

Kucian, K., Loenneker, T., Dietrich, T., Dosch, M., Martin, E., & von Aster, M. (2006). Impaired neural networks for approximate calculation in dyscalculic children: A functional MRI study. *Behavioral and Brain Functions*, *2*(31), 1–17.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic.

Landerl, K., Bevan, A., & Butterworth, B. (2004). Developmental dyscalculia and basic numerical capacities: A study of 8–9 year old students. *Cognition*, *93*, 99–125.

Landerl, K., & Kölle, C. (2009). Typical and atypical development of basic numerical skills in elementary school. *Journal of Experimental Child Psychology*, *103*(4), 546–565.

Liu, X., Wang, H., Corbly, C. R., Zhang, J., & Joseph, J. E. (2006). The involvement of the inferior parietal cortex in the numerical Stroop effect and the distance effect in a two-digit number comparison task. *Journal of Cognitive Neuroscience*, *18*, 1518–1530.

Montessori, M. (1966). *The discovery of the child (M. Johnstone, Trans.)*. Madras: Kalakshetra.

Moyer, R. S., & Landauer, T. K. (1967). Time required for judgments of numerical inequality. *Nature*, *215*, 1519–1520.

National Research Council Committee on Early Childhood Mathematics. (2009). C. T. Cross, T. A. Woods, & H. Schweingruber (Eds.), *Mathematics learning in early childhood: Paths toward excellence and equity*. Washington, DC: National Academies Press.

Noël, M.-P. (2005). Finger gnosia: A predictor of numerical abilities in children? *Child Neuropsychology*, *11*, 413–430.

Nuerk, H.-C., Kaufmann, L., Zoppoth, S., & Willmes, K. (2004). On the development of the mental number line: More, less, or never holistic with increasing age? *Developmental Psychology*, *40*, 1199–1211.

Nuerk, H.-C., Weger, U., & Willmes, K. (2001). Decade breaks in the mental number line? Putting the tens and units back in different bins. *Cognition*, *82*, B25–B33.

Organization for Economic Cooperation and Development (OECD). (2010). *The high cost of low educational performance: The long-run impact of improving PISA outcomes*. Paris: OECD.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. Brighton, UK: Harvester.

Parsons, S., & Bynner, J. (2005). *Does numeracy matter more?* London: National Research and Development Centre for Adult Literacy and Numeracy, Institute of Education.

Piaget, J. (1952). *The child's conception of number*. London: Routledge & Kegan Paul.

Piazza, M., Facoetti, A., Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., Dehaene, S., & Zorzi, M. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition*, *116*(1), 33–41.

Piazza, M., Mechelli, A., Price, C. J., & Butterworth, B. (2006). Exact and approximate judgements of visual and auditory numerosity: An fMRI study. *Brain Research*, *1106*, 177–188.

Pinel, P., Dehaene, S., Rivière, D., & Le Bihan, D. (2001). Modulation of parietal activation by semantic distance in a number comparison task. *NeuroImage*, *14*, 1013–1026.

Price, G. R., Holloway, I., Räsänen, P., Vesterinen, M., & Ansari, D. (2007). Impaired parietal magnitude processing in developmental dyscalculia. *Current Biology*, *17*, 1042–1043.

Qin, Y., Sohn, M.-H., Anderson, J. R., Stenger, V. A., Fissell, K., Goode, A. & Carter, C. S. (2003). Predicting the practice effects on the blood oxygenation level-dependent (BOLD) function of fMRI in a symbolic manipulation task. *Proceedings of the National Academy of Sciences of the United States of America*, *100*, 4951–4956.

Qin, Y., Carter, C. S., Silk, E., Stenger, V. A., Fissell, K., Goode, A., & Anderson, J. R. (2004). The change of the brain activation patterns as children learn algebra equation solving. *Proceedings of National Academy of Sciences of the United States of America*, *101*, 5686–5691.

Räsänen, P., Salminen, J., Wilson, A. J., Aunioa, P., & Dehaene, S. (2009). Computer-assisted intervention for children with low numeracy skills. *Cognitive Development*, *24*, 450–472.

Reigosa-Crespo, V., Valdés-Sosa, M., Butterworth, B., Estévez, N., Rodríguez, M., Santos, E., Torres, P., Suárez, R., & Lage, A. (2011). Basic numerical capacities and prevalence of developmental dyscalculia: The Havana Survey. *Developmental Psychology*. DOI: 10.1037/a0025356

Rivera, S. M., Reiss, S. M., Eckert, M. A., & Menon, V. (2005). Developmental changes in mental arithmetic: Evidence for increased functional specialization in the left inferior parietal cortex. *Cerebral Cortex*, *15*, 1779–1790.

Rosenshine, B. (1995). Advances in research on instruction. *The Journal of Educational Research*, *88*, 262–268.

Rotzer, S., Kucian, K., Martin, E., von Aster, M., Klaver, P., & Loenneker, T. (2008). Optimized voxel-based morphometry in children with developmental dyscalculia. *NeuroImage*, *39*(1), 417–422.

Rousselle, L., & Noël, M.-P. (2007). Basic numerical skills in children with mathematics learning disabilities: A comparison of symbolic vs non-symbolic number magnitude processing. *Cognition*, *102*(3), 361–395.

Rusconi, E., Bueti, D., Walsh, V., & Butterworth, B. (2011). Contribution of frontal cortex to the spatial representation of number. *Cortex*, *47*(1), 2–13.

Rusconi, E., Pinel, P., Dehaene, S., & Kleinschmidt, A. (2010). The enigma of Gerstmann's syndrome revisited: A telling tale of the vicissitudes of neuropsychology. *Brain*, *133*(2), 320–332. DOI: 10.1093/brain/awp281

Rusconi, E., Turatto, M., & Umiltà, C. (2007). Two orienting mechanisms in posterior parietal lobule: An rTMS study of the Simon and SNARC effects. *Cognitive Neuropsychology*, *24*(4), 373–392.

Rusconi, E., Walsh, V., & Butterworth, B. (2005). Dexterity with numbers: rTMS over left angular gyrus disrupts finger gnosis and number processing. *Neuropsychologia*, *43*(11), 1609–1624.

Rykhlevskaia, E., Uddin, L. Q., Kondos, L., & Menon, V. (2009). Neuroanatomical correlates of developmental dyscalculia: Combined evidence from morphometry and tractography. *Frontiers in Human Neuroscience*, *3*(51), 1-13. DOI: 10.3389/neuro.09.051.2009

Sarnecka, B. W., & Gelman, S. A. (2004). Six does not just mean a lot: Preschoolers see number words as specific. *Cognition*, *92*(3), 329–352.

Schmithorst, V. J., & Brown, R. D. (2004). Empirical validation of the triple-code model of numerical processing for complex math operations using functional MRI and group Independent Component Analysis of the mental addition and subtraction of fractions. *NeuroImage*, *22*, 1414–1420.

Sekuler, R., & Mierkiewicz, D. (1977). Children's judgements of numerical inequality. *Child Development*, *48*, 630–633.

Shalev, R. S. (2007). Prevalence of developmental dyscalculia. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 49–60). Baltimore, MD: Brookes.

Shalev, R. S., Manor, O., & Gross-Tsur, V. (2005). Developmental dyscalculia: A prospective six year follow up. *Developmental Medicine and Child Psychology*, *47*(2), 121–125.

Shallice, T., & Evans, M. E. (1978). The involvement of the frontal lobes in cognitive estimation. *Cortex*, *14*, 294–303.

Smith, A. (2004). *Making mathematics count: The report of Professor Adrian Smith's inquiry into post-14 mathematics education*. London: Department for Education and Skills.

Sohn, M.-H., Goode, A., Koedinger, K. R., Stenger, V. A, Fissell, K., Carter, C. S., & Anderson, J. R. (2004). Behavioral equivalence, but not neural equivalence: Neural evidence in mathematical problem solving. *Nature Neuroscience*, *7*, 1193–1994.

Starkey, P., & Cooper, R. G., Jr. (1980). Perception of numbers by human infants. *Science*, *210*, 1033–1035.

Stocco, A., & Anderson, J. R. (2008). Endogenous control and task representation: An fMRI study in algebraic problem solving. *Journal of Cognitive Neuroscience*, *20*, 1300–1314.

Thorndike, E. L. (1922). *The psychology of arithmetic*. New York: Macmillan.

Tsang, J., Blair, K. P., Bofferding, L., Rosenberg-Lee, M., & Schwartz, D. (2011, April). *Educational neuroscience: An example in the context of the integers*. Paper presented at the 2011 Annual Meeting of the American Educational Research Association, New Orleans, LA.

Tzelgov, J., Ganor-Stern, D., & Maymon-Schreiber, K. (2009). The representation of negative numbers: Exploring the effects of mode of processing and notation. *The Quarterly Journal of Experimental Psychology*, *62*, 605–624.

Varma, S., & Schwartz, D. L. (2011). The mental representation of integers: A symbolic to perceptual-magnitude shift. *Cognition*, *121*, 363–385.

Vetter, P., Butterworth, B., & Bahrami, B. (2011). A candidate for the attentional bottleneck: Set-size specific modulation of the right TPJ during attentive enumeration. *Journal of Cognitive Neuroscience*, *23*(3), 728–736. DOI: doi:10.1162/jocn.2010.21472

Von Glasersfeld, E. (1989). Cognition, construction of knowledge, and teaching. *Synthese*, *80*, 121–140.

Vygotsky, L. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Westermann, G., Thomas, M. S. C., & Karmiloff-Smith, A. (2010). Neuroconstructivism. In *The* Wiley-Blackwell *handbook of childhood cognitive development* (pp. 723–748). Oxford: Wiley-Blackwell.

Whitehead, A. N. (1948). *An introduction to mathematics* . London: Oxford University Press. (Original work published 1911)

Williams, P. (2008). *Independent review of mathematics teaching in early years settings and primary schools (final report)*. London: Department for Children, Schools and Families.

Xu, F., & Spelke, E. (2000). Large number discrimination in 6-month-old infants. *Cognition*, *74*, B1–B11.

Zbrodoff, N. J., & Logan, G. D. (2005). What everyone finds: The problem size effect. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 331–345). New York: Psychology.